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Optimal approximate doubles

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Abstract
The nonlocality of quantum states on a bipartite system $\mathcal{A}+\mathcal{B}$ is tested by comparing probabilistic outcomes of two local observables of different subsystems. For a fixed observable A of the subsystem $\mathcal{A}$, its optimal approximate double A' of the other system $\mathcal{B}$ is defined such that the probabilistic outcomes of A' are almost similar to those of the fixed observable A. The case of σ-finite standard von Neumann algebras is considered and the optimal approximate double A' of an observable A is explicitly determined. The connection between optimal approximate doubles and quantum correlations is explained. Inspired by quantum states with perfect correlation, like Einstein–Podolsky–Rosen states and Bohm states, the nonlocality power of an observable A for general quantum states is defined as the similarity that the outcomes of A look like the properties of the subsystem $\mathcal{B}$ corresponding to A'. As an application of optimal approximate doubles, maximal Bell correlation of a pure entangled state on $\mathcal{B}(\mathbb{C}^2)\otimes\mathcal{B}(\mathbb{C}^2)$ is found explicitly.

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1. Introduction

An essential feature of quantum systems is the phenomenon of nonlocality. The first example was provided by the famous Einstein–Podolsky–Rosen (EPR) state [1]. On the EPR state the outcomes of measurements on different local systems are such perfectly correlated that if the outcome of one measurement on a local system is known, then the outcome of some measurement on the other local system can be predicted with certainty. Therefore, the EPR state is said to have perfect correlation. On the other hand, the EPR state [1] suffers a mathematical difficulty that it cannot be represented as a unit vector on the Hilbert space $L^2(\mathbb{R}^2)$.

The first well-defined state having perfect correlation is finite-dimensional and was found by Bohm [2]. Recently the EPR state was formulated as a positive linear functional with norm one on the CCR-algebra $\mathcal{A}(\mathbb{R}^2) \otimes \mathcal{A}(\mathbb{R}^2)$ [3] or on the set of bounded linear operators $\mathcal{B}(L^2(\mathbb{R}^2))$ [4]. One interesting task is then to find out all the states with perfect correlation. The case of finite-dimensional systems was rigorously discussed in [5] and all states with perfect condition on finite-dimensional systems are found and shown to be unitarily equivalent. In [6] the condition of perfect correlation was formulated as a simple equation for general systems. With this equation one may find out all states with perfect correlation on the two-particle system $\mathcal{A}(\mathbb{R}^n) \otimes \mathcal{A}(\mathbb{R}^n)$ [7] which shows a different entanglement property from finite-dimensional systems—there are infinitely many unitarily non-equivalent states with perfect correlation on $\mathcal{A}(\mathbb{R}^n) \otimes \mathcal{A}(\mathbb{R}^n)$.

The condition of perfect correlation is given as follows [6]. Let $\mathcal{A}$ and $\mathcal{B}$ be two commuting von Neumann algebras on a Hilbert space $\mathcal{H}$ and $\mathcal{R}$ the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$. Let ω be a state on $\mathcal{R}$ and A a self-adjoint operator in $\mathcal{A}$. A self-adjoint operator A' in $\mathcal{B}$ is called an EPR double of A with respect to ω if

$$\omega((A - A')^2) = 0. \tag{1}$$

The pair (A, A') may be called an EPR pair with respect to ω and equation (1) is called the perfect correlation condition. Moreover, ω is called an EPR state if every self-adjoint operator A has an EPR double A' with respect to ω, and vice versa.

Consider that ω is a vector state on $\mathcal{R}$ of the following form:

$$\omega(X) = \langle \Omega, X\Omega \rangle \tag{2}$$

with $X \in \mathcal{R}$ where Ω is a unit vector in $\mathcal{H}$. The perfect correlation condition (1) becomes

$$\|A\Omega - A'\Omega\|^2 = 0 \tag{3}$$

with $A = A^* \in \mathcal{A}$ and $A' = A'^* \in \mathcal{B}$. Equation (3) means that the EPR double A' of A can be found just by comparing all vectors $B'\Omega$, $B' \in \mathcal{B}$ to the vector $A\Omega$ and $A'\Omega$ is the closest vector to $A\Omega$ such that $A'\Omega = A\Omega$. Consequently, the main idea of equation (3) is to obtain the optimal approximation of $A\Omega$ from the set $\{B'\Omega\ ;\ B' \in \mathcal{B}\}$. These optimal approximations of all observables $A \in \mathcal{A}$ reveal one important entanglement property of quantum states—the perfect correlation. Following this line, one is interested in optimal approximations of observables for general quantum states.

The concept of EPR doubles may be generalized as follows. For a self-adjoint operator A in $\mathcal{A}$ we define the quantity $q_\omega(A)$ of A:

$$q_\omega(A) = \inf\{\omega((A - B')^2) : B' = B'^* \in \mathcal{B}\}. \tag{4}$$

The quantity $q_\omega(A)$ is a measure of the defect of perfect correlation with respect to A and ω. A self-adjoint operator A' in $\mathcal{B}$ is called an optimal approximate double of A with respect to ω if $\omega((A - A')^2) = q_\omega(A)$. Similarly, we can define $q_\omega(B')$ for a self-adjoint operator B' in $\mathcal{B}$ and the optimal approximate double of B'.

Consider again that ω is a vector state (2). Then $q_\omega(A)$ is determined uniquely by a vector in the closed set $\overline{\{B'\Omega\ ;\ B' = B'^* \in \mathcal{B}\}}$. Furthermore, assume that Ω is separating for $\mathcal{B}$, i.e. $B'\Omega = 0$ with $B' \in \mathcal{B}$ implies $B' = 0$. Then, if the closest vector is of the form $A'\Omega$, A' is unique. Generally it is difficult to find out optimal approximate doubles A' of an observable A for a state ω.

The purpose here is to consider a special class of quantum states that Ω is a cyclic and separating vector for a σ-finite von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$. The observable algebras of two subsystems are given by $\mathcal{A} = \mathcal{M}$ and $\mathcal{B} = \mathcal{M}'$ where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. The state ω on $\mathcal{R}$ is the vector state (2) associated with Ω. In this case Ω is also cyclic and separating for $\mathcal{M}'$ and Ω has the property

$$\overline{\mathcal{M}\Omega} = \mathcal{H} = \overline{\mathcal{M}'\Omega}. \tag{5}$$

Thus, ω is a entangled state. Such states are very interesting in physics. For finite-dimensional systems $\mathbb{C}^n \otimes \mathbb{C}^n$ with observable algebras $\mathcal{M} = \mathcal{B}(\mathbb{C}^n) \otimes I$ and $\mathcal{M}' = I \otimes \mathcal{B}(\mathbb{C}^n)$ we know that pure states ϕ with Schmidt number n,

$$\phi = \sum_{j=1}^{n} \lambda_j |jj\rangle, \qquad \lambda_j > 0, \qquad \sum_{j=1}^{n} \lambda_j^2 = 1, \tag{6}$$

where $\{|j\rangle\}_{j=1}^n$ is an orthonormal basis for $\mathbb{C}^n$, satisfy (5). Other examples are temperature states in quantum statistical mechanics [8]. Furthermore, in local quantum field theory [9] the vacuum state Ω has a much stronger property—the Reeh–Schlieder property. It means that Ω is a cyclic vector for the field algebra $\mathcal{A}(\mathcal{O})$ of any open set $\mathcal{O}$ in the Minkowski space, $\overline{\mathcal{A}(\mathcal{O})\Omega} = \mathcal{H}$.

The quantity $q_\omega(A)$ has a simple geometrical meaning. It is known that AA' is still a self-adjoint operator on $\mathcal{H}$ for a self-adjoint operator $A \in \mathcal{M}$ and a self-adjoint operator $A' \in \mathcal{M}'$. Hence, we have

$$\langle A\Omega, A'\Omega\rangle = \langle \Omega, AA'\Omega\rangle \in \mathbb{R} \tag{7}$$

for $A = A^* \in \mathcal{M}$, $A' = A'^* \in \mathcal{M}'$. Introduce the notation $\mathcal{H}_r$ to view $\mathcal{H}$ as a real Hilbert space by equipping it with the real part of its inner product

$$\langle \xi, \eta\rangle_r = \mathrm{Re}\langle \xi, \eta\rangle$$

for $\xi, \eta \in \mathcal{H}$. Let $\mathcal{M}_s$ and $\mathcal{M}'_s$ denote the subsets of self-adjoint operators of $\mathcal{M}$ and $\mathcal{M}'$. Then $\mathcal{M}_s\Omega$ and $\mathcal{M}'_s\Omega$ are two real subspaces of $\mathcal{H}_r$. Thus, $q_\omega(A)$ is equal to the distance of the vector $A\Omega$ to the closed real subspace $\overline{\mathcal{M}'_s\Omega}$. Hence, searching optimal approximate double $A' \in \mathcal{M}_s$ of a given element $A \in \mathcal{M}_s$ is equal to finding the projection $A'\Omega$ of a vector $A\Omega$ on the closed real subspace $\overline{\mathcal{M}'_s\Omega}$ in $\mathcal{H}_r$. This is the basic idea of our estimation in section 3.

2. Geometrical aspects of standard von Neumann algebras

In this section we first review basic algebraic structures of von Neumann algebras $\mathcal{M}$ with a cyclic and separating vector Ω on a complex Hilbert space $\mathcal{H}$ [10]. Denote $\mathcal{H}_r$ as the real version of the Hilbert space $\mathcal{H}$ as introduced at the end of section 1. Then we construct of a real Hilbert space $\mathcal{X}$ with $\mathcal{H}_r = \mathcal{X} \otimes \mathcal{X}$ and a positive operator A on $\mathcal{X}$ such that $x \in \mathcal{M}_s\Omega$ and $y \in \mathcal{M}'_s\Omega$ can be represented isometrically as $(\xi, A\xi)$ and $(\tilde{\xi}, -A\tilde{\xi})$ with $\xi, \tilde{\xi} \in \mathcal{X}$. This explains the geometrical positions of $\overline{\mathcal{M}_s\Omega}$ and $\overline{\mathcal{M}'_s\Omega}$ on $\mathcal{H}_r$ [11].

Since Ω is cyclic and separating for $\mathcal{M}$, Ω has the property of equation (5). Define the operator S_0 on $\mathcal{H}$ as follows:

$$S_0 : A\Omega \mapsto A^*\Omega, \qquad A \in \mathcal{M}. \tag{8}$$

Due to equation (5) S_0 is closable. Its closure S has the polar decomposition

$$S = J\Delta^{1/2}, \tag{9}$$

with a positive operator Δ and an anti-unitary operator J. Δ and J are called the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \Omega)$, respectively. The Tomita–Takesaki modular theorem [10] says that

$$J\Omega = \Omega = \Delta\Omega \tag{10}$$

and

$$J\mathcal{M}J = \mathcal{M}', \qquad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \qquad t \in \mathbb{R}, \tag{11}$$

where the one-parameter group of automorphisms of $\mathcal{M}$,

$$\sigma_t(A) = \Delta^{it} A \Delta^{-it}, \tag{12}$$

is called the modular group associated with $(\mathcal{M}, \Omega)$. The vector state Ω on $\mathcal{M}$ satisfies the so-called KMS condition that for every pair of elements A, B of $\mathcal{M}$ there is a bounded continuous function $F_{A,B}(z)$ in the strip $0 \leqslant \mathrm{Im}(z) \leqslant 1$ and holomorphic in the interior such that

$$F_{A,B}(t) = \langle \Omega, A\sigma_t(B)\Omega \rangle, \qquad F_{A,B}(t + \mathrm{i}) = \langle \Omega, \sigma_t(B) A\Omega \rangle. \tag{13}$$

This condition defines temperature states in quantum statistical mechanics [8]. The closure of the set

$$\{Aj(A)\Omega \ : \ A \in \mathcal{M}\}$$

with $j(A) = JAJ$ is called the natural positive cone $\mathcal{P}$ associated with $(\mathcal{M}, \Omega)$. Any vector ξ with $J\xi = \xi$ has a unique decomposition $\xi = \xi_1 - \xi_2$, where $\xi_1, \xi_2 \in \mathcal{P}$ and $\xi_1 \perp \xi_2$. $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ is called a standard von Neumann algebra.

From equation (11) we see that $\mathcal{M}_s$ and $\mathcal{M}'_s$ are related by the antilinear *-isomorphism $j : \mathcal{M} \to \mathcal{M}'$ defined by $j(A) = JAJ$. On the other hand, the geometrical positions of $\overline{\mathcal{M}_s\Omega}$ and $\overline{\mathcal{M}'_s\Omega}$ on $\mathcal{H}_r$ are given in [11]: (1) $\mathcal{H}_r$ can be written as the direct of certain real Hilbert spaces, $\mathcal{H}_r = \mathcal{X} \oplus \mathcal{X}$ and $\overline{\mathcal{M}_s\Omega}$ and $\overline{\mathcal{M}'_s\Omega}$ can be represented isomorphically as the graphs of A and $-A$ where A is a positive operator on $\mathcal{X}$; (2) graph(A) and graph$(-A)$ can be rotated isometrically. The constructions of the operator A and the associated Hilbert space $\mathcal{X}$ are basic steps to further analysis in [11]. Pictorially graph(A) and graph$(-A)$ (i.e. $\overline{\mathcal{M}_s\Omega}$ and $\overline{\mathcal{M}'_s\Omega}$) can be viewed as two straight lines L_m and L_{-m} through the origin with opposite slopes m and $-m$ in an XY-plane. It is known that L_m and L_{-m} can be rotated counterclockwise such that L_{-m} is mapped to the x-axis. Then the projection of a point P of L_m on L_{-m} is just the x-component of the new coordinates of P after the rotation. With this picture estimation of the projection of a vector in $\mathcal{M}_s\Omega$ on $\overline{\mathcal{M}'_s\Omega}$ in the next section can be performed with the two steps. First, the projection of a vector of $\overline{\mathcal{M}_s\Omega}$ on $\overline{\mathcal{M}'_s\Omega}$ is equal to the projection of the corresponding vector of graph(A) on graph$(-A)$. Second, the projection of a vector of graph(A) on graph$(-A)$ can be derived with the help of the isomorphism corresponding to the rotation mapping L_{-m} to the x-axis.

Let $\mathcal{K} = \overline{\mathcal{M}_s\Omega}$ and $\widetilde{\mathcal{K}} = \overline{\mathcal{M}'_s\Omega}$ denote the closure of $\mathcal{M}_s\Omega$ and $\mathcal{M}'_s\Omega$, respectively. Clearly, $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are two closed real subspaces of $\mathcal{H}_r$. Let $\coprod$ denote the orthogonality with respect to the real inner product $\langle \xi, \eta \rangle_r$. It was shown [11] that $\mathrm{i}\widetilde{\mathcal{K}}$ is the real orthogonal complement of $\mathcal{K}$: $\mathcal{K}^{\amalg} = \mathrm{i}\widetilde{\mathcal{K}}$ on $\mathcal{H}_r$. Consequently it holds that

$$\mathcal{H}_r = \mathcal{K} \oplus \mathrm{i}\widetilde{\mathcal{K}}. \tag{14}$$

Moreover, due to (8)and (9) we have

$$\Delta^{1/2}|\mathcal{K} = J|\mathcal{K}, \qquad \Delta^{-1/2}|\widetilde{\mathcal{K}} = J|\widetilde{\mathcal{K}}. \tag{15}$$

Thus, $\Delta^{1/2}$ and $\Delta^{-1/2}$ are bounded and invertible on $\mathcal{K}$ and $\widetilde{\mathcal{K}}$.

Let $\mathcal{H}^\natural$ denote the (closed) real eigenspace of J corresponding to the eigenvalue 1,

$$\mathcal{H}^\natural = \mathcal{P} - \mathcal{P} = \{\xi \in \mathcal{H} : J\xi = \xi\}. \tag{16}$$

Since the restriction of the inner product on $\mathcal{H}$ to $\mathcal{H}^\natural$ is real, we use the notation $\mathcal{H}_r^\natural$ to remind us that $\mathcal{H}^\natural$ is a real Hilbert space. Because of $\mathcal{H}_r^\natural \coprod \mathrm{i}\mathcal{H}_r^\natural$ and $\mathcal{H} = \mathcal{H}^\natural + \mathrm{i}\mathcal{H}^\natural$, it holds that

$$\mathcal{H}_r = \mathcal{H}_r^\natural \oplus \mathrm{i}\mathcal{H}_r^\natural. \tag{17}$$

From equation (11) it follows that $J\mathcal{K} = \widetilde{\mathcal{K}}$ and $J\widetilde{\mathcal{K}} = \mathcal{K}$. As a consequence, $\mathcal{H}_r^\natural$ can be obtained from $\mathcal{K}$ or $\widetilde{\mathcal{K}}$. Let $q = \frac{1}{2}(I + J)$ be the real projection onto $\mathcal{H}_r^\natural$. Because of equation (15) q has different representations on $\mathcal{K}$ and $\widetilde{\mathcal{K}}$. Define

$$Q = \tfrac{1}{2}(I + J)|\mathcal{K} = \tfrac{1}{2}(I + \Delta^{1/2})|\mathcal{K}, \tag{18}$$

$$\widetilde{Q} = \tfrac{1}{2}(I + J)|\widetilde{\mathcal{K}} = \tfrac{1}{2}(I + \Delta^{-1/2})|\widetilde{\mathcal{K}}. \tag{19}$$

It was shown [11] that Q and $\widetilde{Q}$ have the same range, i.e.

$$Q(\mathcal{K}) = \widetilde{Q}(\widetilde{\mathcal{K}}) = \mathcal{H}_r^\natural \tag{20}$$

and both Q and $\widetilde{Q}$ are invertible. Therefore, elements of both $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ can be represented in terms of the elements of $\mathcal{H}_r^\natural$ [11].

Lemma 1. *Let $(I - \Delta^{1/2})/(I + \Delta^{1/2})$ has the following polar decomposition:*

$$\frac{I - \Delta^{1/2}}{I + \Delta^{1/2}} = UA \tag{21}$$

with a positive operator A and an unitary operator U:

$$A = \frac{|I - \Delta^{1/2}|}{I + \Delta^{1/2}}, \qquad U = p_1 - p_2 \tag{22}$$

where p_1 is the spectral projection of Δ corresponding to the open interval $(0, 1)$ and p_2 is the spectral projection of Δ corresponding to the open interval $(1, \infty)$. Then it holds that

$$\mathcal{K} = \left\{\xi + UA\xi : \xi \in \mathcal{H}_r^\natural\right\}, \qquad \widetilde{\mathcal{K}} = \left\{\xi - UA\xi : \xi \in \mathcal{H}_r^\natural\right\}. \tag{23}$$

Clearly, the positive operator A is bounded, $0 \leqslant A \leqslant I$. In terms of A and U the inverse of Q and $\widetilde{Q}$ can be represented by

$$Q^{-1} = (I + UA)|\mathcal{H}_r^\natural \qquad \text{and} \qquad \widetilde{Q}^{-1} = (I - UA)|\mathcal{H}_r^\natural. \tag{24}$$

Thus it holds that

$$\widetilde{Q}^{-1}Q = \Delta^{1/2}|\mathcal{K} = J|\mathcal{K}, \tag{25}$$

$$Q^{-1}\widetilde{Q} = \Delta^{-1/2}|\widetilde{\mathcal{K}} = J|\widetilde{\mathcal{K}}. \tag{26}$$

Due to $J\Delta J = \Delta^{-1}$ we have $JA = AJ$. Therefore, A maps $\mathcal{H}_r^\natural$ into $\mathcal{H}_r^\natural$.

Define the graph of a bounded operator X on a Hilbert space $\mathcal{X}$ to be the subset graph(X) of $\mathcal{X} \oplus \mathcal{X}$ with

$$\text{graph}(X) = \{(\xi, A\xi) \ : \ \xi \in \mathcal{X}\}.$$

Moreover, let (L_1, N_1) be a pair of (closed) real subspaces of the real Hilbert space $\mathcal{H}_1$, and (L_2, N_2) be a correspondence pair of the real Hilbert space $\mathcal{H}_2$. (L_1, N_1) is said to be (isometrically) equivalent to (L_2, N_2), denoted by $(L_1, N_1) \cong (L_2, N_2)$, if there exists an isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that $V(L_1) = L_1$ and $V(N_1) = N_2$.

Let $X = A$ be the operator given by equation (22) and $\mathcal{X} = \mathcal{H}_r^\natural$. Then we may identify graph(A) and graph$(-A)$ with $\mathcal{K}$ and $\widetilde{\mathcal{K}}$, respectively [11].

Theorem 2. *There exists an operator $A : \mathcal{H}_r^\natural \to \mathcal{H}_r^\natural, 0 \leqslant A \leqslant I$ such that $(\text{graph}(A), \text{graph}(-A)) \cong (\mathcal{K}, \widetilde{\mathcal{K}})$, i.e. there exists an isometry V from the real Hilbert space $\mathcal{H}_r^\natural \oplus \mathcal{H}_r^\natural$ onto $\mathcal{H}_r$ such that*

$$V(\text{graph}(A)) = \mathcal{K} \quad \textit{and} \quad V(\text{graph}(-A)) = \widetilde{\mathcal{K}}. \tag{27}$$

The operator A is unique up to isometric equivalence, i.e. if B is another operator with the same properties as A, then there is an isometry $W : \mathcal{H}_r^\natural \to \mathcal{H}_r^\natural$ such that $B = WAW^*$.

Specially, A can be chosen as

$$A = \frac{|I - \Delta^{1/2}|}{I + \Delta^{1/2}}\bigg|\mathcal{H}_r^\natural. \tag{28}$$

In the following, the isometry V for the special choice (28) of A is explained. Let $\mathcal{C}$ denote the set of fixed points of the modular group σ_t:

$$\mathcal{C} = \{A \in \mathcal{M} | \sigma_t(A) = A\}. \tag{29}$$

By the KMS-boundary condition (13) it was shown [11] that

$$\mathcal{C} = \{A \in \mathcal{M} | \omega(AB) = \omega(BA),\ \forall\, B \in \mathcal{M}\}.$$

Moreover, it holds [11] that

$$\overline{\mathcal{C}_s\Omega} = \mathcal{K} \cap \widetilde{\mathcal{K}} = \mathcal{K} \cap \mathcal{H}_r^\natural = \widetilde{\mathcal{K}} \cap \mathcal{H}_r^\natural \tag{30}$$

where $\mathcal{C}_s$ is the set of self-adjoint operators in $\mathcal{C}$. Denote by p_0 the projection onto the subspace $\overline{\mathcal{C}\Omega}$ of invariant vectors. The mapping V for the special choice (28) of A is given by

$$V : \mathcal{H}_r^\natural \oplus \mathcal{H}_r^\natural \to \mathcal{H}_r, \qquad V(\xi \oplus \eta) = \xi + U'\eta \tag{31}$$

with $U' = \mathrm{i}p_0 + U = \mathrm{i}p_0 + p_1 - p_2$, where p_1 and p_2 are the projections defined in lemma 1. Since $Jp_0 = p_0 J$ and $p_2 = Jp_1 J$, we have $JU' = -U'J$ and hence U' maps $\mathcal{H}_r^\natural$ onto $\mathrm{i}\mathcal{H}_r^\natural$. Together with (17) V is an isometry from $\mathcal{H}_r^\natural \oplus \mathcal{H}_r^\natural$ onto $\mathcal{H}_r$ $\left(=\mathcal{H}_r^\natural \oplus \mathrm{i}\mathcal{H}_r^\natural\right)$. In particular, for vectors in graph(A) and graph$(-A)$ it holds

$$V(\xi, A\xi) = \xi + U'A\xi = \xi + UA\xi = Q^{-1}\xi \in \mathcal{K}, \tag{32}$$

$$V(\xi, -A\xi) = \xi - U'A\xi = \xi - UA\xi = \widetilde{Q}^{-1}\xi \in \widetilde{\mathcal{K}}. \tag{33}$$

because of lemma 1 and equation (24).

There is a simple picture to demonstrate the relation of graph(A) and graph$(-A)$. Suppose both the x-axis and y-axis of an XY-plane represent the real Hilbert space $\mathcal{H}_r^\natural$. Then the XY-plane represents the Hilbert space $\mathcal{H}_r^\natural \oplus \mathcal{H}_r^\natural$ (and hence $\mathcal{H}_r$). The origin represents the set $\overline{\mathcal{C}_s\Omega}$ due to equation (30). Furthermore, the graphs of operator A and $-A$, graph(A) and graph$(-A)$, are represented as two straight lines L_m and L_{-m} through the origin with slope m and $-m$, $m > 0$, respectively. It reflects the symmetric positions of $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ with respect to $\mathcal{H}_r^\natural$.

This simple picture can also demonstrate that graph(A) and graph$(-A)$ can be isometrically rotated. In the XY-plane the two lines L_m and L_{-m} can be rotated counterclockwise with the angle θ, $\theta = \arctan m$, so that the line L_{-m} representing graph$(-A)$ is mapped to the x-axis while the line L_m representing graph(A) to the line $L_{m'}$ with slope $m' = \tan 2\theta$. Correspondingly, graph(A) and graph$(-A)$ are isometrically equivalent to graph(T) and $\mathcal{H}_r^\natural \oplus \{0\}$ for some positive operator T,

$$(\text{graph}(A), \text{graph}(-A)) \cong (\text{graph}(T), \mathcal{H}^\natural \oplus \{0\}). \tag{34}$$

This isometric equivalence is implemented by the isometry $V' : \mathcal{H}_r^\natural \oplus \mathcal{H}_r^\natural \to \mathcal{H}_r^\natural \oplus \mathcal{H}_r^\natural$:

$$V'((\xi, \eta)) = \left(\frac{I}{(I + A^2)^{1/2}}\xi - \frac{A}{(I + A^2)^{1/2}}\eta,\ \frac{A}{(I + A^2)^{1/2}}\xi + \frac{I}{(I + A^2)^{1/2}}\eta\right) \tag{35}$$

and it holds that

$$V'(\text{graph}(A)) = \text{graph}(T), \qquad V'(\text{graph}(-A)) = \mathcal{H}_r^\natural \oplus \{0\} \tag{36}$$

with $T = 2A/(I - A^2)$. Consequently, the projection of a vector in graph(A) onto graph$(-A)$ can be found by using V' and T.

3. Optimal approximate doubles

Theorem 3. *Let $x \in \mathcal{K}$. Then there exists a unique element $\tilde{x} \in \widetilde{\mathcal{K}}$ such that $\|x - \tilde{x}\| = \min_{\tilde{y}\in\widetilde{\mathcal{K}}} \|x - \tilde{y}\|$ with $\tilde{x}$ given by*

$$\tilde{x} = PJx \qquad \textit{with} \quad P = \frac{2}{\Delta^{-1/2} + \Delta^{1/2}} \tag{37}$$

where Δ and J are the modular operator and the modular conjugation, respectively. We may also call $\tilde{x}$ as the optimal approximate double of x.

Proof. As mentioned before, we work with the real Hilbert space $\mathcal{H}_r$ and $\mathcal{K} = \overline{\mathcal{M}_s\Omega}$ and $\widetilde{\mathcal{K}} = \overline{\mathcal{M}_s'\Omega}$ are two closed real subspaces of the real Hilbert space $\mathcal{H}_r$. It holds that

$$\|x - \tilde{y}\| = \|x - \tilde{y}\|_r, \tag{38}$$

where $\|\cdot\|_r$ is the norm induced by the inner product of $\mathcal{H}_r$.

Our estimation consists of two steps: (i) x and $\tilde{y}$ are represented as the elements of graph(A) and graph$(-A)$, respectively; (ii) graph(A) and graph$(-A)$ are isometrically transformed onto graph(T) and $\mathcal{H}_r^\natural \oplus \{0\}$, respectively. So the estimation can be performed with ease.

(i) By theorem 2, x and $\tilde{y}$ can be represented by vectors in graph(A) and graph$(-A)$. In particular, there exist unique $\xi \in \mathcal{H}_r^\natural$ and $\tilde{\xi} \in \mathcal{H}_r^\natural$ such that

$$\|x - \tilde{y}\|_r = \|(\xi, A\xi) - (\tilde{\xi}, (-A)\tilde{\xi})\|_r \qquad \text{with} \quad \xi = Qx, \quad \tilde{\xi} = \widetilde{Q}\tilde{y} \tag{39}$$

by equations (32) and (33).

(ii) Because of the isometric equivalence $(\text{graph}A, \text{graph}(-A)) \cong (\text{graph}T, \mathcal{H}^\natural \oplus \{0\})$ with $T = 2A/(I - A^2)$ equation (39) becomes

$$\|(\xi, A\xi) - (\tilde{\xi}, (-A)\tilde{\xi})\|_r = \|(\xi_1, T\xi_1) - (\xi_2, 0)\|_r \tag{40}$$

where $\xi_1, \xi_2 \in \mathcal{H}_r^\natural$ are given uniquely by

$$\xi_1 = \frac{1 - A^2}{(1 + A^2)^{1/2}}\xi \qquad \text{and} \qquad \xi_2 = (1 + A^2)^{1/2}\tilde{\xi}. \tag{41}$$

due to equations (34)–(36). Thus, the minimum of $\|x - \tilde{y}\|$ is achieved iff $\xi_2 = \xi_1$, i.e. iff

$$\tilde{\xi} = P\xi, \qquad \text{with} \quad P = \frac{1 - A^2}{1 + A^2}. \tag{42}$$

Because of $JA = AJ$ we also have $JP = PJ$. Thus, P maps $\mathcal{H}_r^\natural$ into $\mathcal{H}_r^\natural$. Moreover, P can be represented in terms of the modular operator Δ:

$$P = \frac{2}{\Delta^{-1/2} + \Delta^{1/2}}. \tag{43}$$

For $A' \in M_s'$ we have

$$PA'\Omega = \int_{\mathbb{R}} \frac{2\,\mathrm{d}t}{\mathrm{e}^{\pi t} + \mathrm{e}^{-\pi t}}\,\sigma_t(A')\Omega. \tag{44}$$

Due to the modular theorem (11) we have $\sigma_t(A_2')\Omega \in \widetilde{\mathcal{K}}$ and hence $PA_2'\Omega \in \widetilde{\mathcal{K}}$. Thus, $P\widetilde{\mathcal{K}} \subset \widetilde{\mathcal{K}}$. On the other hand, due to equation (24) it holds $\widetilde{Q}^{-1}P\xi = P\widetilde{Q}^{-1}\xi \in \widetilde{\mathcal{K}}$ for all $\xi \in \mathcal{H}_r^\natural$. Consequently, the minimum is achieved if and only if

$$\tilde{y} = \tilde{Q}^{-1}\tilde{\xi} = \tilde{Q}^{-1}PQx = P\tilde{Q}^{-1}Qx = PJx. \tag{45}$$

The last equality follows from equation (25). $\square$

Corollary 4. *Let $A = A^* \in \mathcal{M}$. Then the optimal approximate double $A' = A'^* \in \mathcal{M}'$ of A can be uniquely given by $A' = \mathcal{P}(A)$ with*

$$\mathcal{P}(A) = 2I_\lambda(JAJ) \qquad \textit{with} \quad \lambda = 1 \tag{46}$$

where I_λ is given in [10],

$$I_\lambda(x) = \lambda^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}t \frac{\lambda^{it}}{\mathrm{e}^{\pi t} + \mathrm{e}^{-\pi t}} \sigma_t(x). \tag{47}$$

The function $\mathcal{P}(\cdot)$ mapping A to its optimal approximate double $\mathcal{P}(A)$ is a one-to-one mapping with norm equal to one.

Proof. From equation (11) the operator A' given by equation (46) is in $\mathcal{M}'$. Furthermore, it holds that

$$\begin{aligned} A'\Omega &= 2\int_{-\infty}^{\infty} \frac{1}{\mathrm{e}^{\pi t} + \mathrm{e}^{-\pi t}} \Delta^{it}(JAJ)\Delta^{-it}\Omega \\ &= 2\int_{-\infty}^{\infty} \frac{1}{\mathrm{e}^{\pi t} + \mathrm{e}^{-\pi t}} \Delta^{it} JA\Omega \\ &= \frac{2}{\Delta^{1/2} + \Delta^{-1/2}} JA\Omega. \end{aligned}$$

Since Ω is separating for $\mathcal{M}'$, A' is determined uniquely and hence $\mathcal{P}(\cdot)$ is one-to-one. □

The case $\Delta = I$ corresponds to EPR states on which we have $P = I$ and all observables $A \in \mathcal{M}_s$ are perfectly correlated with their optimal approximate doubles $A' \in \mathcal{M}_s'$ given by $A' = JAJ$. The following corollary says which observables are perfectly correlated for general states.

Corollary 5. *The following are equivalent.*

(1) $x = \tilde{x}$
(2) $\|\tilde{x}\| = \|x\|$
(3) $x \in \overline{\mathcal{C}_s\Omega}$

Proof. The implication of 2 from 1 is trivial. If $\|\tilde{x}\| = \|x\|$, Jx is an eigenvector of Δ with eigenvalue 1. Therefore, $\Delta^{-1/2}Jx = Jx$. Due to $J|\mathcal{K} = \Delta^{1/2}|\mathcal{K}$ we have $x = \Delta^{1/2}x$ and thus $x \in \overline{\mathcal{C}_s\Omega}$.

On the other hand, if $x \in \overline{\mathcal{C}_s\Omega}$, we have $\Delta x = x$ and $Px = x$. Moreover, $Jx = \Delta^{1/2}x = x$. Thus $\tilde{x} = PJx = Px = x$. □

Hence, observables A are perfectly correlated with their optimal approximate doubles A' iff observables A are in $\mathcal{C}_s$. In this case, we have $A' = JAJ$.

4. Nonlocality power

The optimal approximate double $\tilde{x} = PJx$ of a vector $x \in \mathcal{K}$ consists of two operators J and P. The operator J comes from the mapping $\kappa(A) = JA^*J$ which maps $\mathcal{M}$ to $\mathcal{M}'$ bijectively due to (11). For the case that $\Delta = I$ (i.e. $P = I$) it follows from equations (10) and (11) that $A\Omega = \kappa(A)\Omega$ and $A^*\Omega = (\kappa(A))^*\Omega$ for any $A \in \mathcal{M}$. Hence,

$$A\Omega = \kappa(A)\Omega, \qquad \text{with} \quad A \in \mathcal{M}_s, \quad \kappa(A) \in \mathcal{M}_s'. \tag{48}$$

It means that the probabilistic outcomes of an observable A of one subsystem $\mathcal{M}$ are the same as those of the observable $\kappa(A)$ of the other subsystem $\mathcal{M}'$—which is just the perfect correlation considered by EPR. Consequently $\tilde{x} = x$ and $J = I$ on $\mathcal{K}$. Furthermore, equation (48) implies that observables of both subsystems give the same probabilistic outcomes. Thus, we have $\mathcal{M}_s\Omega = \mathcal{M}_s'\Omega$ and $\mathcal{K} = \widetilde{\mathcal{K}}$.

For the case that $\Delta \neq I$ (i.e. $P \neq I$) it follows from corollary 5 that $\|\tilde{x}\| < \|x\|$ for some $x \in \mathcal{K}$. It follows that $\mathcal{K} \neq \widetilde{\mathcal{K}}$. Thus, there is some observable $A \in \mathcal{M}_s$ which is not perfectly correlated with any observable $A' \in \mathcal{M}_s'$. But there is still a symmetry between $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ given by the anti-unitary mapping $A\Omega \to \kappa(A)\Omega$, $A \in \mathcal{M}_s$, i.e. $x \to Jx$, $x \in \mathcal{K}$. Clearly, the operator P causes the contraction of the norm from $\|x\|$ to $\|\tilde{x}\|$. Hence, for the vector state Ω without perfect correlation P describes how an observable of one subsystem is correlated with observables of the other subsystem.

Recall that in probability theory and statistics the correlation coefficient $\mathrm{cor}(X, Y)$ between two random variables X and Y with expectation values μ_X and μ_Y and standard deviations σ_X and σ_Y is defined as

$$\mathrm{cor}(X, Y) = \frac{E((X - \mu_X)(Y - \mu_Y))}{\sigma_X \sigma_Y}, \tag{49}$$

where E is the expectation value operator. It measures the strength and direction of a linear relationship between the X and Y variables. The correlation coefficient $\mathrm{cor}(X, Y)$ always takes a value between -1 and 1, with 1 (or -1) indicating a perfect positive (or negative) linear relationship. If there is no linear correlation or a weak linear correlation between X and Y, then $\mathrm{cor}(X, Y)$ is close to 0.

For an observable X in a vector state Ω of a quantum system the expectation value operator is given by $E(X) = \langle \Omega, X\Omega \rangle$. Consider two local observables $A \in \mathcal{M}_s$ and $B' \in \mathcal{M}_s'$. If $E(A) = 0 = E(B')$ (i.e. $\mu_A = 0 = \mu_{B'}$), then we have

$$\mathrm{cor}(A, B') = \frac{\langle \Omega, AB'\Omega \rangle}{\|A\Omega\| \cdot \|B'\Omega\|} = \frac{\langle A\Omega, B'\Omega \rangle}{\|A\Omega\| \cdot \|B'\Omega\|} = \cos\theta \tag{50}$$

where θ is the angle between the vectors $A\Omega$ and $B'\Omega$. Hence, the cosine of the angle between the vectors $A\Omega$ and $B'\Omega$ is a measure of the correlation of the outcomes of A and B'. Generally we may say that $A\Omega$ and $B'\Omega$ encode probabilistic outcomes of A and B' and the cosine of the angle between $A\Omega$ and $B'\Omega$ is a measure of the correlation of probabilistic outcomes of A and B'.

Since the sign of cosine is irrelevant to nonlocality, we are interested in the following quantity:

$$\cos^2\theta = \frac{|\langle A\Omega, B'\Omega \rangle|^2}{\|A\Omega\|^2 \cdot \|B'\Omega\|^2}. \tag{51}$$

Due to theorem 3 we have

$$|\langle A\Omega, B'\Omega \rangle|^2 = |\mathrm{Re}\, \langle PJA\Omega, B'\Omega \rangle|^2 \tag{52}$$

and the quantity (51) has the maximal value $p(A)$ with

$$p(A) = \frac{\|PJA\Omega\|^2}{\|A\Omega\|^2} \tag{53}$$

if and only if $B'\Omega$ is parallel to $PJA\Omega$. Thus, the quantity (51) is maximal iff B' is proportional to the optimal approximate double A' of A, i.e. $B' = \alpha A'$, $\alpha \in \mathbb{R}$, with $A' = \mathcal{P}(A)$ defined as (46). Since $PJ = JP$ and J is an anti-unitary operator, it holds that

$$p(A) = \frac{\|PA\Omega\|^2}{\|A\Omega\|^2}. \tag{54}$$

The value of $p(A)$ is between 0 and 1. Due to corollary 5 an observable $A \in \mathcal{M}$ with $p(A) = 1$ is perfectly correlated with its optimal double $A' = \mathcal{P}(A) = JAJ \in \mathcal{M}'$, while an observable $A \in \mathcal{M}$ with $p(A) < 1$ cannot be perfectly correlated to any observable $B' \in \mathcal{M}'$. More precisely, the probabilistic outcomes of a local observable A of the subsystem $\mathcal{M}$ are correlated with the probabilistic outcomes of observables B' of the other subsystem $\mathcal{M}'$ with a correlation coefficient in the range $-\sqrt{p(A)} \leqslant \mathrm{cor}(A, B') \leqslant \sqrt{p(A)}$.

Furthermore, by (51) and (52) the operator P characterizes the similarity between $A\Omega$ and $B'\Omega$ and $p(A)$ gives the highest degree of similarity between $A\Omega$ and $B'\Omega$ when they are normalized. In particular, $A\Omega/\|A\Omega\|$ is similar to $A'\Omega/\|A'\Omega\|$ by $p(A)$. Moreover, the projection of $A\Omega$ on the unit vector $A'\Omega/\|A'\Omega\|$ is $A'\Omega$. Thus, we may say that $A\Omega$ looks like $A'\Omega$ with similarity $p(A)$.

It is well known that the probabilistic outcomes of an observable A of the local subsystem $\mathcal{M}$ are said to be properties of $\mathcal{M}$. Thus, we may say that $A\Omega$ encodes properties of the local subsystem $\mathcal{M}$ obtained by applying the observable $A \in \mathcal{M}$. For a bipartite system $\mathcal{M} + \mathcal{M}'$ in a entangled state Ω, it is interesting to see that the properties of the subsystem $\mathcal{M}'$ can be obtained without applying observables B' of $\mathcal{M}'$ directly. Consider the situation that two subsystems $\mathcal{M}$ and $\mathcal{M}'$ are separated by some distance and they share a cyclic and separating state Ω in common. We want to know the properties of $\mathcal{M}'$, but for some reason we cannot apply observables B' of $\mathcal{M}'$. If Ω is perfectly correlated, then by applying an observable A of $\mathcal{M}$ the probabilistic outcomes of A can be interpreted as the properties of $\mathcal{M}'$ corresponding to its optimal doubles A' due to $A'\Omega = A\Omega$. This is just what EPR states and Bohm states tell us. On the other hand, if Ω is not perfectly correlated, then by applying observables of $\mathcal{M}$ we can still get the properties of $\mathcal{M}'$ by chance. It is because the projection of $A\Omega$ on $\widetilde{\mathcal{K}}$ is $A'\Omega$ and the similarity between $A\Omega$ and $A'\Omega$ is $p(A)$. We may say that $A\Omega$ is interpreted as $A'\Omega$ with probability $p(A)$. It means that we can obtain the properties of the subsystem $\mathcal{M}'$ corresponding to A' with probability $p(A)$. In this aspect nonlocality of Ω is demonstrated by the acquirement of properties of $\mathcal{M}'$ by observables of $\mathcal{M}$ and we may call $p(A)$ the nonlocality power of the local operators A with respect to Ω.

Consider an example of type I factors. Let $\mathcal{M}$ and $\mathcal{M}'$ denote the operator algebras $\mathcal{M} = \mathcal{B}(\mathbb{C}^n) \otimes \mathbb{I}$ and $\mathcal{M}' = \mathbb{I} \otimes \mathcal{B}(\mathbb{C}^n)$ and ω be a vector state Ω of the form $\Omega = \sum_1^n \rho_j^{1/2} |e_j f_j\rangle$ with $\rho_j > 0$ and $\sum_1^n \rho_j = 1$ where $\{e_j\}$ and $\{f_j\}$ are two orthonormal bases of $\mathbb{C}^n$. The modular operator and the modular conjugation are given by

$$J\left(\sum \lambda_{jk} |e_j f_k\rangle\right) = \sum \overline{\lambda_{jk}} |e_k f_j\rangle, \tag{55}$$

$$\Delta = \sum \frac{\rho_j}{\rho_k} |e_j\rangle\langle e_j| \otimes |f_k\rangle\langle f_k|. \tag{56}$$

Due to corollary 5, self-adjoint operators in $\mathcal{C}_s$ have the maximal nonlocality power of 1. For self-adjoint operators not in $\mathcal{C}_s$ we consider two typical self-adjoint operators $P_\psi = |\psi\rangle\langle\psi|$ with $|\psi\rangle = (1/\sqrt{2})(|e_m\rangle + |e_l\rangle)$ and $A_{ml} = |e_m\rangle\langle e_l| + |e_l\rangle\langle e_m|, m \neq l$. The probabilistic outcomes of $P_\psi \otimes I$ and $A_{ml} \otimes I$ while applying to Ω are encoded as

$$(P_\psi \otimes I)\Omega = \tfrac{1}{2}\left(\rho_m^{1/2}|e_m f_m\rangle + \rho_l^{1/2}|e_m f_l\rangle + \rho_m^{1/2}|e_l f_m\rangle + \rho_l^{1/2}|e_l f_l\rangle\right), \tag{57}$$

$$(A_{ml} \otimes I)\Omega = \rho_l^{1/2}|e_m f_l\rangle + \rho_m^{1/2}|e_l f_m\rangle. \tag{58}$$

Thus, the optimal approximate doubles of $(P_\psi \otimes I)\Omega$ and $(A_{ml} \otimes I)\Omega$ are

$$PJ(P_\psi \otimes I)\Omega = \frac{1}{2}\left(\rho_m^{1/2}|e_m f_m\rangle + \frac{2\rho_l\rho_m^{1/2}}{\rho_m + \rho_l}|e_l f_m\rangle + \frac{2\rho_l^{1/2}\rho_m}{\rho_l + \rho_m}|e_m f_l\rangle + \rho_l^{1/2}|e_l f_l\rangle\right), \tag{59}$$

$$PJ(A_{ml}\otimes I)\Omega = \frac{2\rho_l\rho_m^{1/2}}{\rho_l+\rho_m}|e_l f_m\rangle + \frac{2\rho_l^{1/2}\rho_m}{\rho_m+\rho_l}|e_m f_l\rangle \tag{60}$$

and the optimal approximate doubles of P_ψ and A_{ml} are given by

$$\mathcal{P}(P_\psi\otimes I) = I\otimes\frac{1}{2}\left(|f_m\rangle\langle f_m| + \frac{2\rho_m^{1/2}\rho_l^{1/2}}{\rho_m+\rho_l}|f_m\rangle\langle f_l| + \frac{2\rho_m^{1/2}\rho_l^{1/2}}{\rho_m+\rho_l}|f_l\rangle\langle f_m| + |f_l\rangle\langle f_l|\right) \tag{61}$$

$$\mathcal{P}(A_{ml}\otimes I) = I\otimes\frac{2\rho_m^{1/2}\rho_l^{1/2}}{\rho_m+\rho_l}(|f_m\rangle\langle f_l| + |f_l\rangle\langle f_m|). \tag{62}$$

We see that the optimal approximate double $\mathcal{P}(P_\psi\otimes I)$ of a one-dimensional projection P_ψ cannot be represented as $I\otimes P_\phi$ for some vector ϕ generally. On the other hand, the optimal approximate double of $A_{ml}\otimes I$ is its image under the mapping $\kappa(X) = JX^*J$ with some coefficient. Moreover, it holds that

$$\|(P_\psi\otimes I)\Omega\|^2 = \frac{1}{2}(\rho_m+\rho_l), \tag{63}$$

$$\|PJ(P_\psi\otimes I)\Omega\|^2 = \frac{1}{4}\left(\rho_m+\rho_l+\frac{4\rho_m\rho_l}{\rho_m+\rho_l}\right), \tag{64}$$

$$\|(A_{ml}\otimes I)\Omega\|^2 = (\rho_m+\rho_l), \tag{65}$$

$$\|PJ(A_{ml}\otimes I)\Omega\|^2 = \frac{4\rho_m\rho_l}{\rho_m+\rho_l}. \tag{66}$$

Although $\|(P_\psi\otimes I)\Omega\| < \|(A_{ml}\otimes I)\Omega\|$, we have $\|PJ(P_\psi\otimes I)\Omega\| > \|PJ(A_{ml}\otimes I)\Omega\|$. The nonlocality power p_ψ of P_ψ is larger then the nonlocality power p_{ml} of A_{ml}. More precisely, we have

$$p(P_\psi\otimes I) = \frac{1}{2}\left(1+\frac{4\rho_m\rho_l}{(\rho_m+\rho_l)^2}\right) \qquad \text{and} \qquad p(A_{ml}\otimes I) = \frac{4\rho_m\rho_l}{(\rho_m+\rho_l)^2}. \tag{67}$$

In summary, by applying $P_\psi\otimes I$ and $A_{ml}\otimes I$ to Ω the probabilistic outcomes are encoded by $(P_\psi\otimes I)\Omega$ and $(A_{ml}\otimes I)\Omega$ whose projections on $\widetilde{\mathcal{K}}$ are given by $PJ(P_\psi\otimes I)\Omega$ and $PJ(A_{ml}\otimes I)\Omega$, respectively. We say that $(P_\psi\otimes I)\Omega$ and $(A_{ml}\otimes I)\Omega$ are interpreted as $PJ(P_\psi\otimes I)\Omega$ and $PJ(A_{ml}\otimes I)\Omega$ with probability $p(P_\psi\otimes I)$ and $p(A_{ml}\otimes I)$ and thus properties of $\mathcal{M}'$ corresponding to $\mathcal{P}(P_\psi\otimes I)$ and $\mathcal{P}(A_{ml}\otimes I)$ are obtained with probabilities $p(P_\psi\otimes I)$ and $p(A_{ml}\otimes I)$. Moreover, from (67) it follows that $P_\psi\otimes I$ gives more precise properties of the system $\mathcal{M}'$ than $A_{ml}\otimes I$ and we may conclude that the superposition of $|e_m\rangle$ and $|e_l\rangle$ causes more nonlocal effect than A_{ml}.

5. Bell correlation

Another measure of nonlocal effects can be given by Bell correlation which is linear with respect to observables. Let $\mathcal{A}$ and $\mathcal{B}$ be two independent C^*-algebras and ω is a state on $\mathcal{A}\otimes\mathcal{B}$. We call (A_1, A_2, B_1', B_2') an admissible quadruple if $A_i\in\mathcal{A}$ and $B_i'\in\mathcal{B}$ with $-1\leqslant A_i, B_i'\leqslant 1$ for $i=1,2$. Bell correlation of an admissible quadruple (A_1, A_2, B_1', B_2') with respect to ω is given [12] by

$$\beta(\omega; A_1, A_2, B_1', B_2') = \omega(A_1B_1' + A_1B_2' + A_2B_1' - A_2B_2'). \tag{68}$$

We say that Bell's inequality is satisfied for an admissible quadruple (A_1, A_2, B_1', B_2') if

$$|\beta(\omega; A_1, A_2, B_1', B_2')| \leqslant 2. \tag{69}$$

If Bell's inequality is violated, then a local hidden variable model of the correlation is not allowed [13].

Consider the case that $\mathcal{A} = \mathcal{M}$, $\mathcal{B} = \mathcal{M}'$ and ω is given by a cyclic and separating vector Ω for $\mathcal{M}$ and $\mathcal{M}'$. Bell's correlation can be written in the form

$$\beta(\omega; A_1, A_2, B_1', B_2') = \langle A_1\Omega, B_1'\Omega\rangle + \langle A_1\Omega, B_2'\Omega\rangle + \langle A_2\Omega, B_1'\Omega\rangle - \langle A_2\Omega, B_2'\Omega\rangle. \tag{70}$$

Let A_1' and A_2' be optimal approximate doubles of A_1 and A_2. Bell correlation (70) becomes

$$\beta(\omega; A_1, A_2, B_1', B_2') = \langle A_1'\Omega, B_1'\Omega\rangle_r + \langle A_1'\Omega, B_2'\Omega\rangle_r + \langle A_2'\Omega, B_1'\Omega\rangle_r - \langle A_2'\Omega, B_2'\Omega\rangle_r. \tag{71}$$

It holds that

$$|\beta(\omega; A_1, A_2, B_1', B_2')| \leqslant \|A_1'\Omega + A_2'\Omega\|_r \, \|B_1'\Omega\|_r + \|A_1'\Omega - A_2'\Omega\|_r \, \|B_2'\Omega\|_r, \tag{72}$$

where $\|\cdot\|_r$ is the norm introduced by the real inner product $\langle\cdot,\cdot\rangle_r = \mathrm{Re}\,\langle\cdot,\cdot\rangle$. Thus, $\beta(\omega; A_1, A_2, B_1', B_2')$ has the maximal absolute value $\beta(\omega; A_1, A_2)$ with

$$\beta(\omega; A_1, A_2) = \max_{-1\leqslant B_1', B_2'\leqslant 1} |\beta(\omega; A_1, A_2, B_1', B_2')| \tag{73}$$

$$= \alpha_1 \|A_1'\Omega + A_2'\Omega\|_r^2 + \alpha_2 \|A_1'\Omega - A_2'\Omega\|_r^2. \tag{74}$$

$\beta(\omega; A_1, A_2, B_1', B_2')$ is maximal if and only if B_1' and B_2' is taken to be $B_1' = \alpha_1(A_1' + A_2')$ and $B_2' = \alpha_2(A_1' - A_2')$ where α_1, α_2 are positive factors such that $\|B_1'\| = \|B_2'\| = 1$. We may call $\beta(\omega; A_1, A_2)$ the maximal Bell correlation of A_1 and A_2 in the state ω.

Consider a pure state Ω in $\mathbb{C}^2 \otimes \mathbb{C}^2$ of the form

$$|\Omega\rangle = \rho_0^{1/2}|e_0 f_0\rangle + \rho_1^{1/2}|e_1 f_1\rangle \tag{75}$$

such that $\{|e_0\rangle, |e_1\rangle\}$ and $\{|f_0\rangle, |f_1\rangle\}$ are two orthonormal bases of $\mathbb{C}^2$ and $\rho_j > 0$ with $\rho_0 + \rho_1 = 1$. Let $\mathcal{M} = \mathcal{B}(\mathbb{C}^2) \otimes \mathbb{I}$ and $\mathcal{M}' = \mathbb{I} \otimes \mathcal{B}(\mathbb{C}^2)$. Thus, $(\mathcal{M}, \Omega)$ is a finite standard von Neumann algebra.

As usual, Pauli matrices is denoted by $\sigma_x, \sigma_y, \sigma_z$. Assume $\{|e_0\rangle, |e_1\rangle\}$ and $\{|f_0\rangle, |f_1\rangle\}$ are eigenvectors of σ_z in two local systems corresponding to eigenvalue 0 and 1. We see that the three vectors $(\sigma_a \otimes I)\Omega, a = x, y, z$, are pairwise orthogonal in the real Hilbert space $(\mathbb{C}^2 \otimes \mathbb{C}^2)_r$ and so are their optimal approximate doubles $PJ(\sigma_a \otimes I)\Omega, a = x, y, z$:

$$\tilde{s}_x = PJ(\sigma_x \otimes I)\Omega = 2\sqrt{\rho_0\rho_1}\left(\rho_0^{1/2}|e_0 f_1\rangle + \rho_1^{1/2}|e_1 f_0\rangle\right), \tag{76}$$

$$\tilde{s}_y = PJ(\sigma_x \otimes I)\Omega = 2\mathrm{i}\sqrt{\rho_0\rho_1}\left(\rho_0^{1/2}|e_0 f_1\rangle - \rho_1^{1/2}|e_1 f_0\rangle\right), \tag{77}$$

$$\tilde{s}_z = PJ(\sigma_x \otimes I)\Omega = \rho_0^{1/2}|e_0 f_0\rangle - \rho_1^{1/2}|e_1 f_1\rangle. \tag{78}$$

It follows that $\|\tilde{s}_x\| = \|\tilde{s}_y\| = 2\sqrt{\rho_0\rho_1} \leqslant 1$ and equals 1 if and only if $\rho_0 = \rho_1 = 1/2$. Moreover. the optimal approximate doubles of Pauli matrices are then given by

$$\mathcal{P}(\sigma_x \otimes I) = I \otimes 2\sqrt{\rho_0\rho_1}\sigma_x, \tag{79}$$

$$\mathcal{P}(\sigma_y \otimes I) = I \otimes -2\sqrt{\rho_0\rho_1}\sigma_y, \tag{80}$$

$$\mathcal{P}(\sigma_z \otimes I) = I \otimes \sigma_z. \tag{81}$$

Therefore, we get the following maximal Bell correlations:

$$\beta(\Omega; I, \sigma_z) = 2, \tag{82}$$

$$4(\sqrt{2}-1) \leqslant \beta(\Omega; I, \sigma_a) = \frac{2(1+4\rho_0\rho_1)}{1+2\sqrt{\rho_0\rho_1}} < 2, \qquad a = x, y \tag{83}$$

$$0 < \beta(\Omega; \sigma_x, \sigma_y) = 4\sqrt{2\rho_0\rho_1} \leqslant 2\sqrt{2}, \tag{84}$$

$$2 < \beta(\Omega; \sigma_z, \sigma_a) = 2(1+4\rho_0\rho_1)^{1/2} \leqslant 2\sqrt{2}, \qquad a = x, y. \tag{85}$$

$\beta(\omega; I, \sigma_z)$ corresponds to the case of classical communication. $\beta(\omega; \sigma_{x,}\sigma_y)$ is between 0 and $2\sqrt{2}$ where 0 corresponds to the limiting cases of pure product states while $2\sqrt{2}$ corresponds to the maximally entangled states. $\beta(\omega; \sigma_x, \sigma_y) \leqslant \beta(\omega; \sigma_z, \sigma_a)$, $a = x, y$ and all the three are equal when they have the maximal value $2\sqrt{2}$ if and only if $\rho_0 = \rho_1 = 1/2$. Bell's inequality is satisfied for pairs (I, σ_x), (I, σ_y) and (I, σ_z) since they are commutative.

Furthermore, we define the maximal Bell correlation of two independent C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ in the state ω as

$$\beta(\omega; \mathcal{A}, \mathcal{B}) = \max_{-1 \leqslant A_1, A_2 \leqslant 1} \beta(\omega; A_1, A_2). \tag{86}$$

Thus, Bell's inequality is violated in the state ω if $\beta(\omega; \mathcal{A}, \mathcal{B}) > 2$. Let A_1 has eigenvalues λ_i. From the linearity of the Bell correlation $\beta(A_1, A_2, B_1, B_2)$ with respect to λ_i it follows that $\beta(\omega; A_1, A_2)$ reaches the maximal value only if the absolute values of all eigenvalues A_1 are one, $\|\lambda_i\| = 1$ for all i. Thus, to find $\beta(\omega, \mathcal{A}, \mathcal{B})$ it is sufficient to consider A_1 and A_2 with eigenvalues 1 or -1.

For the two-dimensional system $\mathbb{C}^2$ self-adjoint operators with eigenvalues 1 or -1 are $I, -I$ and the traceless self-adjoint matrices H with norm one,

$$H = a\sigma_x + b\sigma_y + c\sigma_z, \qquad \text{with} \quad a, b, c \in \mathbb{R}, \quad a^2 + b^2 + c^2 = 1. \tag{87}$$

From (79) to (81) the optimal approximate doubles of $H \otimes I$ and $(H \otimes I)\Omega$ are given by

$$\begin{aligned} H' &= \mathcal{P}(H \otimes I) = I \otimes (2\sqrt{\rho_0\rho_1}(a\sigma_x - b\sigma_y) + c\sigma_z), \\ H'\Omega &= PJ(H \otimes I)\Omega = a\tilde{s}_x + b\tilde{s}_y + c\tilde{s}_z, \end{aligned} \tag{88}$$

respectively. One observes that

$$\|H'\| = \|H'\Omega\|. \tag{89}$$

If the vectors $H'\Omega$ are written as $H'\Omega = x\tilde{e}_x + y\tilde{e}_y + z\tilde{e}_z$ with orthonormal vectors $\tilde{e}_x = \tilde{s}_x/\|\tilde{s}_x\|$, $\tilde{e}_y = \tilde{s}_y/\|\tilde{s}_y\|$, and $\tilde{e}_z = \tilde{s}_z$ in $(\mathbb{C}^2 \otimes \mathbb{C}^2)_r$, then

$$\frac{x^2}{4\rho_0\rho_1} + \frac{y^2}{4\rho_0\rho_1} + z^2 = 1. \tag{90}$$

Let H_1 and H_2 be the matrices of the form (87) and H_1' and H_2' their optimal approximate doubles. From (72) and (89) it follows that

$$\beta(\Omega; H_1, H_2) \leqslant \|(H_1' + H_2')\Omega\|_r + \|(H_1' - H_2')\Omega\|_r \tag{91}$$

$$\leqslant \sqrt{2}\big(\|(H_1' + H_2')\Omega\|_r^2 + \|(H_1' - H_2')\Omega\|_r^2\big)^{1/2} \tag{92}$$

$$= 2\big(\|H_1'\Omega\|_r^2 + \|H_2'\Omega\|_r^2\big)^{1/2}. \tag{93}$$

The second inequality (92) follows from Cauchy–Schwarz inequality. The equality in (92) holds if and only if $\|(H_1' + H_2')\Omega\|_r = \|(H_1' - H_2')\Omega\|_r$, i.e. if and only if $H_1'\Omega$ and $H_2'\Omega$ are orthogonal in $(\mathbb{C}^2 \otimes \mathbb{C}^2)_r$. The equality (93) follows from the parallelogram law.

Since $\|H'\Omega\|$ given by (88) is symmetric with respect to a and b, it is sufficient to consider matrices of the form $H = a\sigma_x + c\sigma_z$, $a, c \in \mathbb{R}$, $a^2 + c^2 = 1$. Consequently, finding

$\beta(\Omega; \mathcal{M}, \mathcal{M}')$ can be done by maximizing the sum of squares of the lengths of two orthogonal vectors $H_1'\Omega$ and $H_2'\Omega$ on $x^2/(4\rho_0\rho_1) + z^2 = 1$. Explicit calculations show

$$\max \beta(\Omega; H_1, H_2) = 2(1 + 4\rho_0\rho_1)^{1/2}, \tag{94}$$

which happens if $H_1 = \sigma_x$ and $H_2 = \sigma_z$, for example. We come to the conclusion that

$$\beta(\omega; \mathcal{M}, \mathcal{M}') = 2(1 + 4\rho_0\rho_1)^{1/2} \tag{95}$$

which is always greater than 2. Thus, Bell's inequality is violated and quantum correlations in the state Ω cannot explained by local hidden variable models.

6. Concluding summary and discussions

The great idea of Einstein, Podolsky and Rosen is reconsidered in this work. The nonlocality of a bipartite system $\mathcal{A} + \mathcal{B}$ is tested by comparing the outcomes of local observables on both subsystems. For a fixed observable A of one subsystem $\mathcal{A}$ we define an optimal approximate double A' of the other system $\mathcal{B}$ whose probabilistic outcomes are most similar to those of the fixed observable A. Perfect correlation is an extreme case of such comparison. If for any observable A there exists an observable A' whose probabilistic outcomes are the same as those of A, and vice versa, then the state is called perfectly correlated. Well-known examples of perfect correlation are EPR states for continuous systems and Bohm states for finite dimensional systems. In this case A' is an optimal approximate double of A and it can be said that by applying the observable A of the subsystem $\mathcal{A}$ the properties of the subsystem $\mathcal{B}$ corresponding to the optimal approximate doubles A' is revealed. This leads to the interesting question: what if quantum states are not perfectly correlated?

Here we consider a special class of entangled states including pure states on finite systems, temperature states and the vacuum state. The observable algebras of two subsystems are given by a von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ on a The Hilbert space $\mathcal{H}$ and the entangled state ω is the vector state associated with a unit vector $\Omega \in \mathcal{H}$ which is cyclic and separating for both $\mathcal{M}$ and $\mathcal{M}'$. Our results provide detailed comparisons of probabilistic outcomes of local measurements on the state Ω and the optimal approximate double A' of an observable A is determined uniquely. One essential point in our method is that the Hilbert space is taken with a real inner product. It makes our estimation straightforward. The reason is as follows. For $A = A^* \in \mathcal{M}$ and $B = B^* \in \mathcal{M}'$ we see that AB' is still a self-adjoint operator and hence $\langle A\Omega, B'\Omega\rangle = \langle \Omega, AB'\Omega\rangle$ is real. Consequently, comparisons of vectors $A\Omega$ and $B'\Omega$ can be performed with a simple projection method. The optimal approximate vector $A'\Omega$ to the fixed vector $A\Omega$ can be found by the projection on the closed real space spanned by vectors $B'\Omega$, $B \in \mathcal{M}'$. Then the optimal approximate double A' of A can be determined and we have $\langle A\Omega, B'\Omega\rangle = \mathrm{Re}\langle A'\Omega, B'\Omega\rangle$. It is interesting to note that the inner product of $A\Omega$ and $B'\Omega$ is equal to the quantum correlation of A and B' which plays an essential role in modern quantum information theory.

The physical meaning of optimal approximate doubles will be clearer when together with the nonlocality power $p(A)$ of A. We know that probabilistic outcomes of an observable $A \in \mathcal{M}$ are said to be the properties of $\mathcal{M}$. As demonstrated by EPR states and Bohm states, if two subsystems $\mathcal{M}$ and $\mathcal{M}'$ are perfectly correlated, then properties of $\mathcal{M}'$ can be obtained exactly by virtue of applying observables of $\mathcal{M}$. For vector states Ω without perfect correlation we define the nonlocality power $p(A)$ of an local observable $A \in \mathcal{M}$ as the similarity that $A\Omega$ looks like the properties $A'\Omega$ of the subsystem $\mathcal{M}'$ corresponding to A'. More precisely, when applying a local measurement $A \in \mathcal{M}$ to Ω, the probabilistic outcomes are encoded as a vector $A\Omega$ whose projection on $\tilde{\mathcal{K}}$ is given by $A'\Omega$ with $\widetilde{\mathcal{K}} = \overline{\mathcal{M}_s'\Omega}$. We say that the probabilistic outcome vector $A\Omega$ of A is interpreted as the probabilistic outcome vector $A'\Omega$ of A' with probability $p(A)$. It means that the properties of the subsystem $\mathcal{M}'$ can be acquired by chance by applying the observables of $\mathcal{M}$. Consequently, nonlocality of quantum states can be described by the following question: how many properties of the subsystem $\mathcal{M}'$ can be obtained by observables of the subsystem $\mathcal{M}$?

The value of $p(A)$ is between 0 and 1, $0 \leqslant p(A) \leqslant 1$. Clearly for states with perfect correlation like EPR states or Bohm states we have $p(A) = 1$ for all $A \in \mathcal{M}$. With a smaller value of $p(A)$ we get less accurate properties of the subsystem $\mathcal{M}'$. Examples in section 4 show that nonlocality power can be enhanced by superposition.

One application of optimal approximate doubles is to find Bell's correlation. Bell's inequality is such an important step in quantum theory that it makes nonlocality of quantum states experimentally testable. The experimental data used in Bell's inequality are just quantum correlations. For any (A_1, A_2, B_1', B_2') of observables Bell's correlation can be given by (A_1', A_2', B_1', B_2') where A_1', A_2' are optimal approximate doubles of A_1, A_2. Then estimating the maximum of Bell's correlation can be performed easily. As an example, maximal Bell's correlation for a pure entangled state $\phi = a|00\rangle + \sqrt{1-a^2}|11\rangle, 0 < a < 1$ on a two-dimensional systems is found explicitly and shown to be larger than 2 for $0 < a < 1$. Therefore, local hidden variable models are not suitable for any entangled pure state ϕ.

It is worth noting that optimal approximate doubles are defined for any quantum state. Let $\mathcal{A}$ and $\mathcal{B}$ be two independent C^*-algebras and ω a quantum state on $\mathcal{A} \otimes \mathcal{B}$. One can always find a Hilbert space $\mathcal{H}_\omega$ with a mapping $\pi : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}(\mathcal{H}_\omega)$ and a unit vector $\Omega \in \mathcal{H}_\omega$ such that Ω is cyclic for $\pi(\mathcal{A} \otimes \mathcal{B})$ and $\omega(X) = \langle \Omega, \pi(X)\Omega \rangle$ with $X \in \mathcal{A} \otimes \mathcal{B}$ [10]. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras generated by $\pi(\mathcal{A})$ and $\pi(\mathcal{B})$. In $\mathcal{H}_\omega$ the geometrical meaning of optimal approximate double $A' \in \mathcal{N}$ of an observable $A \in \mathcal{M}$ still holds: $A'\Omega$ is the optimal approximate vector to $A\Omega$ among $\overline{\{B'\Omega\ ;\ B' = B'^* \in \mathcal{N}\}}$. Generally we have $\overline{\mathcal{M}\Omega} \neq \mathcal{H}_\omega \neq \overline{\mathcal{N}\Omega}$. Thus, equation (5) does not hold in general. But the projection method can be still applied. In particular, one may hope that some characterizations of separable states can be given in terms of optimal approximation doubles.

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